Anomalous transport and quantum-classical correspondence

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We present evidence that anomalous transport in the classical standard map results in strong enhancement of fluctuations in the localization length of quasienergy states in the corresponding quantum dynamics. This generic effect occurs even far from the semiclassical limit and reflects the interplay of local and global quantum suppression mechanisms of classically chaotic dynamics. Possible experimental scenarios are also discussed. [S1063-651X(99)03006-8]

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It is generally accepted that quantum mechanics suppresses chaotic classical motion. Numerous studies have identified mechanisms for suppression which, for our purposes, fall into two broad classes. One class is exemplified by ''dynamical localization'' where quantum eigenstates are localized in a variable such as momentum despite the deterministic diffusion seen in the limiting classical dynamics [1]. This effect is analogous to Anderson localization in tight-binding models [2] and, in one-dimension, the localized wave function has a characteristic exponential form. The scale is the localization length ξ *which is related to the classical diffusion constant*, an important fact for the work reported here. There is also a time scale t^* beyond which quantum and classical dynamics deviate. Despite a few counterexamples, dynamical localization provides a ''global'' mechanism for quantum suppression.

The second class can be motivated by the fast deviation of wave packet dynamics from the classical motion even in the semiclassical limit. This effect is contained in the logarithmic break time $\tau_\hbar = \ln(I/\hbar)/\Gamma$ for the breakdown of quantum-classical correspondence [3], relative to a characteristic Lyapunov exponent Γ and action I. However, the quantum dynamics retains features of the classical dynamics for times beyond this estimate. ''Scarring'' [4,5] refers to quantum coherences associated with local, unstable and marginally stable, classical invariant structures. Though many open questions remain, this effect has been numerically and experimentally observed in a variety of strongly coupled quantum systems.

We present evidence of another example of ''classical persistence'' due to anomalous diffusion and accelerator modes in chaotic dynamics. There has been considerable activity in the area of anomalous classical transport [6–8]. Of particular relevance to the present work is that even in an established paradigm such as the standard map, described by $q_{n+1} = q_n + p_{n+1}$ and $p_{n+1} = p_n + K \sin q_n$, surprising new results were reported.

In the limit of large K, the classical standard map dynamics is diffusive in the action variable p with diffusion constant $\langle p^2 \rangle / t = K^2/2 \equiv D_{ql}$. What is also established is that away from this limit, the diffusion constant exhibits peaks with changing K and varies in time [8]. Recent results [6,7] with improved precision and detail show that there are values of K where the effective diffusion constant (over some time) can be different from D_{ql} by many orders of magnitude. Figure 1 shows these windows of anomalous diffusion which are not very narrow and exhibit substructure in each of the peaks.

It was shown that there is no diffusion at all and that the random walk process corresponds to Lévy-like wandering with a transport exponent μ given by $\langle p^2 \rangle \approx t^\mu$, where $\mu > 1$. μ varies with K and has the ''normal'' value $\mu = 1$ only at special values of K. Superdiffusion occurs for $\mu > 1$ leading to anomalous growth in momentum. It was explicitly shown [6] that this could result from an island hierarchy with a peculiar topological structure near ''accelerator'' modes which appear at special values of K. Near these values, classical trajectories stick to the boundaries of these islands which leads to flights of arbitrary length. The net result is strong intermittency and superdiffusion.

One signature of anomalous transport is constructed from the set of recurrence times $\{\tau_j\}$, $j = 0,1, \ldots$, for a classical trajectory originating in a region of phase space. The prob-

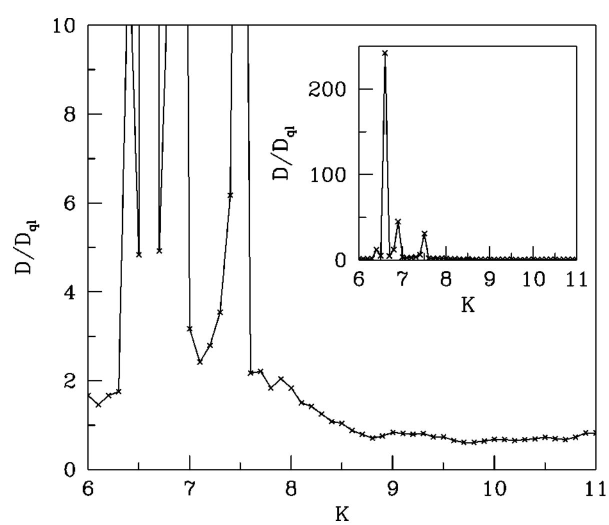

FIG. 1. Diffusion coefficient D for the standard map, normalized to $D_{ql} = K^2/2$, as a function of K. The deviations from the quasilinear prediction span a wide window of K and, as seen from the inset, can be large at select values of K, even far from the bifurcation point $K = 2\pi$.

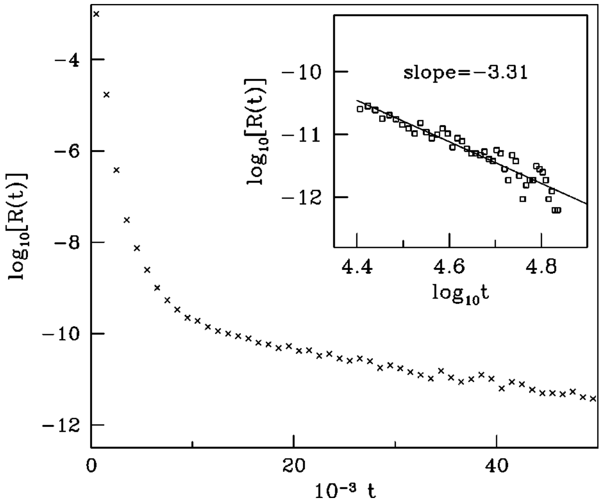

FIG. 2. Distribution of Poincaré cycles. The inset clearly displays the power-law tail of the distribution.

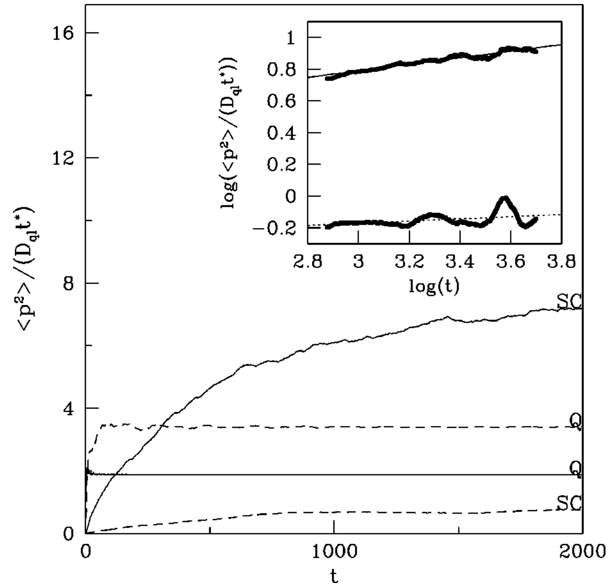

FIG. 3. Variation of $\langle p^2 \rangle$ (normalized to the saturation value $D_{ql}t^*$) with the number of kicks t. Two representative values of a, one large (Q,$a \approx 0.61803399$) and the other small (SC,$a \approx 0.042340526$), are shown for $K=11$ (dashed lines) and $K^* = 6.905$ (solid lines). Note that the larger a value was time-averaged to remove fluctuations that mask the trends. The inset shows a log-log plot (at longer times, ≈ 5000 kicks) for $a = 0.042340526$, $K^* = 6.905$ (upper curve), and $K = 11.0$ (lower curve). The corresponding slopes are 0.21 (solid line) and 0.06 (dashed line) respectively.

ability distribution $R(t)$ of Poincaré cycles $\{t_j\} = \{\tau_{j+1} - \tau_j\}$ can now be computed. For ''perfect'' mixing and normal diffusion, $R(t)$ is strictly Poissonian whereas it exhibits powerlike asymptotics for anomalous kinetics. This classical characteristic of anomalous transport is illustrated in Fig. 2, and is important for the corresponding quantum dynamics as well.

We pose several questions, arising from classical anomalous diffusion, in the corresponding quantum dynamics of the standard map, or, equivalently, the ''delta-kicked rotor'' $H = p^2/2 + K \cos q \Sigma_n \delta(t-n)$. The scale length ξ is related to the classical diffusion constant and even follows the variations in D with K [11]. Thus, we anticipate that strong enhancement in D resulting from anomalous diffusion should also be reflected in ξ. The nature of dynamical localization in these special windows is a related question. Further, as small local structures in the classical phase space lead to the superdiffusion, these issues directly pertain to the interplay of local and global aspects of quantum suppression. This idea is reinforced by the enhanced diffusion that ought to be visible in the quantum dynamics even far from the semiclassical limit, where scarring is significant. Other authors have considered quantum dynamics in the presence of anomalous transport [9,10] though in a different regime. We focus on the situation where the quantization scale $2\pi\hbar$ is much larger than the size of the islands. Thus, any coherences associated with the islands would be due to ''scarred'' quantum states.

Consider the evolution of $\langle p^2 \rangle$ with time t (measured as the number of kicks) in the quantum dynamics for two values of the parameter K. The computation uses fast-Fourier transforms to evolve a plane wave initial state under repeated application of the single-kick evolution operator

$$U = \exp(-ip^2/2\hbar)\exp[-iK\cos(q)/\hbar]. \quad (1)$$

The expectation values are then computed from the time-evolved wave function. $K_c = 6.908745\cdots$ is a critical value [6] in a wide parametric window dominated by anomalous transport. We illustrate this regime by considering $K^* = 6.905$ and contrast it with $K=11$, where the effective diffusion coefficient is actually below D_{ql} (seen in Fig. 1). $\hbar = 2\pi a$, where several irrational values of a in the range $0.04-1$ were considered. In the quasilinear limit, the quantum diffusion is expected to saturate at $t^* = \alpha D_{ql}/\hbar^2$ where α was numerically shown to be 1/2 [1,11]. ξ measured in angular momentum quanta is $\approx t^*$. Figure 3 shows $\langle p^2 \rangle/(D_{ql}t^*)$ as a function of time for two representative values of a. For large a, both values of K result in strong saturation with $K=11$ having a higher saturation value than $K^* = 6.905$. However, with decreasing a the enhanced diffusion for $K^* = 6.905$ becomes evident and there is a slowdown in growth in lieu of saturation (over the time considered). By contrast, $K=11$ appears to saturate at a value which is about an order of magnitude lower. This difference is considerably larger than estimates based on quasilinear analysis. The inset in Fig. 2 verifies the existence of power-law behavior in $\langle p^2 \rangle$ *for longer times* at K^* as compared with saturation at $K=11$. Note that the slope points to subdiffusive rather than to superdiffusive growth for which we have no clear explanation at present.

A stronger signature appears on considering the variation of localization length ξ with K. It was shown [11] that ξ tracks the variations in the classical diffusion coefficient D when plotted with respect to $K_q = K\sin(\hbar/2)/(\hbar/2)$, an important correction at larger $\hbar$. Figure 4 shows the variation in ξ, normalized to the quasilinear estimate, with respect to K_q. ξ is obtained by fitting an exponential to the time-dependent wave function for long times. The peaks in the inset coincide with the classical accelerator modes at $K = 2\pi$ and 4π and ξ computed at different times reflect the same features. The detailed scan displays a reasonable correspondence with the

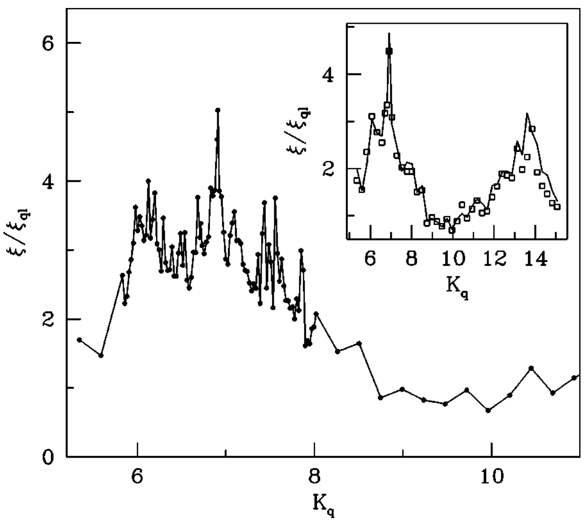

FIG. 4. Oscillations in the localization length ξ normalized to the quasilinear estimate as a function of K_q. $a \approx 0.131267463$ which makes the corrections to the classical stochasticity parameter K significant. The inset shows peaks associated with the accelerator modes while the main figure zooms in on the region considered in Fig. 1. The inset also shows ξ computed after 3000 kicks (points) as well as 9000 (line) kicks. Note that the wave function averaged over 600 kicks is used to determine ξ.

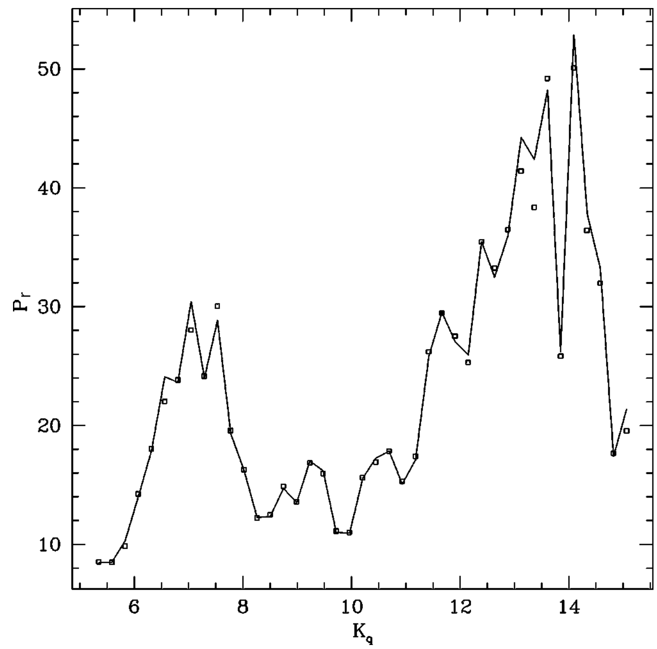

FIG. 6. Participation ratio as a function of K_q. Time averages of the autocorrelation function over 1000(points) and 6000(line) kicks are shown. $a \approx 0.131267463$.

results shown in Fig. 1, despite being far from the semiclassical limit. It should be noted that the highest value occurs at $K_q \approx K_c$ defined earlier.

We now consider the localized wave function when the classical transport is predominantly anomalous. Figure 5 shows the evolved momentum distribution averaged over 600 kicks. Panel (a) corresponds to K_q at a peak in Fig. 4 while (b) sits in the valley. Localization does occur though the characteristic line shape is strongly nonexponential for K_q in the neighborhood of the peaks. The contrast with the clearly exponential line shape shown in (b) is striking. The central region in (a) also displays curvature and an exponential fit yields a ξ which is considerably larger than the quasilinear prediction. However, in case (b), the computed ξ is well approximated by the quasilinear estimate. Moving away from the peak in K_q decreases the prominence of the *shoulders* in the line shape. The shoulders develop over a time (which depends on $\hbar$) which is consistent with the crossover time from exponential to power-law behavior in Fig. 1.

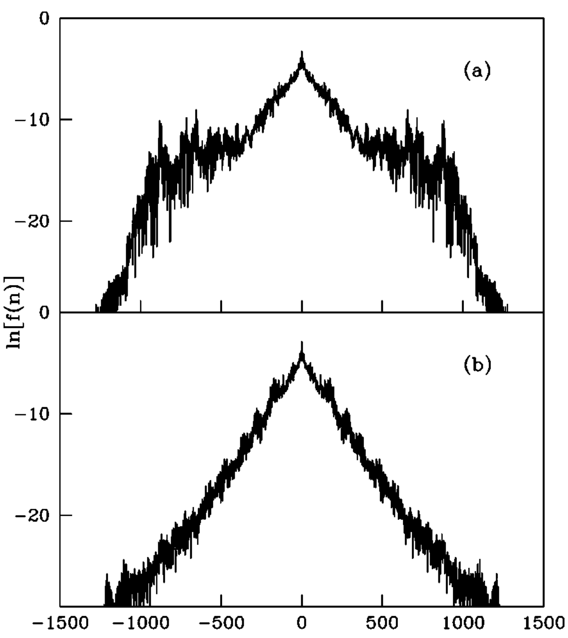

FIG. 5. Line shapes after 6000 kicks starting from a plane wave initial condition. (a) corresponds to $K_q \approx K_c = 6.90845\cdots$ while (b) corresponds to $K_q = 10.69$. $a \approx 0.131267463$ and the noise in the line shapes was reduced by averaging over 600 kicks.

Arguments for deviations of the localization length from the simple estimates proceed from the long-time averaged momentum distribution $\langle f_n \rangle = \Sigma_m |\psi_m(n_0)|^2 |\psi_m(n)|^2$, starting from a plane wave $n = n_0$ initial condition. $\psi_m(n)$ is the plane wave representation of the eigenfunction with quasienergy ω_m. The analysis of localization assumes that all quasienergy states are exponentially localized and that the fluctuations in ξ are small. If this were true in the case of anomalous transport, the oscillations in Fig. 4 should directly correlate with the number of quasienergy states participating in the dynamics. To address this view, we computed the participation ratio P_r from the time averaged autocorrelation function

$$P_r = \lim_{T \to \infty} \frac{1}{T} |\langle \psi(T) | \psi(0) \rangle|^2. \tag{2}$$

As seen from Fig. 6, the only correlation with Fig. 4 occurs in the valley where the line shape is exponentially localized. At the peaks, these results strongly support the idea that anomalous transport results in strong fluctuations in ξ among the individual quasienergy states. This was verified by constructing the ‘‘entire’’ spectrum of U under conditions necessary for dynamical localization. The average value of ξ

does not reflect the large changes in D though the dispersion $\Delta\xi$ exhibits large fluctuations. This view is being explored further at present.

Theoretically, the classical flights of arbitrary length result (for the standard map) from a hierarchy of boundary layer islands [6]. The island area S_n, associated period T_n and Lyapunov exponent Γ_n scale as $S_n=\lambda_S^n S_0$, $T_n=\lambda_T^n T_0$, and $\Gamma_n=\lambda_\Gamma^n \Gamma_0$ with the generation n of the island chain. $\lambda_S<1$, $\lambda_T>1$, and $\lambda_\Gamma \approx 1/\lambda_T$.

Quantum flights are possible as long as $S_n \geqslant \hbar$ leading to a critical value $n_\hbar=|\ln \hbar/S_0|/|\ln \lambda_S|$. The corresponding time is $T_\hbar=T_0(S_0/\hbar)^{1/\mu}$, where $\mu=|\ln \lambda_S|/\ln \lambda_T|>1$ is a classical transport exponent [6]. We generalize the break time $\tau_\hbar$ defined earlier to $\ln(I/\hbar)/\Gamma_n$ where I was some characteristic action. The longest time corresponds to the smallest $\Gamma_n=\Gamma_\hbar=1/T_\hbar$ from which

$$\tau_\hbar=T_\hbar \ln(I/\hbar)=T_0(S_0/\hbar)^{1/\mu}\ln(I/\hbar). \quad (3)$$

The result points to power-law dependence on $\hbar$ as well as a possible crossover in scaling behavior. Note that $T_\hbar$ for flights is a pure quantum effect, necessary for localization, which cannot occur if arbitrarily long flights exist with high probability (not exponentially small). Theoretical estimates are considerably harder when $\hbar>S_0$.

The issue of fluctuations in ξ is relevant to an atom optics realization of the kicked pendulum [12]. Here, the localization properties are similar to those in the quantum standard map, though small islands persist in the classical dynamics. As we have illustrated, this could lead to strong fluctuations in the localized wave function. More recent experiments have realized the δ-kicked rotor system [13] where the measurement of diffusion, over time scales longer than those predicted earlier, can provide direct evidence of any anomalous transport. We note that signatures of anomalous transport are similar to those resulting from external noise. This may be especially relevant in the weak noise limit and will be considered in detail elsewhere.

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